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Generalised Backlund transformation for some non-linear partial differential-difference equations

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Abstract. In this paper we determine the generalised Bäcklund transformations connecting two different solutions of non-linear partial differential-difference equations solvable by an inverse method. The scattering problem for the inverse method is the discretised scalar Schrödinger equation. In particular we isolate the Bäcklund transformations relating two different solutions of the same equation and a transformation between a solution of the same equation at two different times.

1. Introduction

Flaschka (1974) made the first successful application of the inverse scattering transform to solving a non-linear differential-difference equation in his solution of a one-dimensional exponential lattice with infinite length, the Toda lattice (Toda 1967a, b, 1970). The corresponding scattering problem was the discretised Schrödinger equation for which Flaschka extended to the whole line the analysis of the inverse problem for the discretised Schrödinger equation on the half line given by Case and Kac (1973) and Case (1973). The Toda lattice has been the subject of intensive investigation in recent years (see *Prog. Theor. Phys. Suppl. No.* 59 1976 for many references) and has played a comparable role to the Korteweg–de Vries equation in the study of the soliton phenomenon in non-linear differential-difference equations. It is interesting to note in particular that the Korteweg–de Vries equation is one of the possible continuum limits of the Toda lattice (Toda and Wadati 1973).

In this paper we closely follow the ideas of Ablowitz *et al* (1974) as developed by them in their paper on the generalised Zakharov–Shabat scattering problem (Zakharov and Shabat 1972). They emphasise the relationship between the inverse scattering transform and Fourier analysis and the important role played by the dispersion relation of the linearised evolution equation. Indeed for a given linear evolution equation (one spatial dimension) they show that its dispersion relation determines the corresponding solvable non-linear evolution equation. Fundamental to this approach is the scattering problem satisfied by the 'squared eigenfunctions' of the original scattering problem (Ablowitz *et al* 1974, Kaup 1976). Flaschka and McLaughlin (1976) have already applied this method using the discretised Schrödinger equation as the original scattering problem and identified the class of solvable non-linear differential-difference equations. The Toda lattice is the simplest member of this class.

In deriving the formulae for the generalised Bäcklund transformations and other functional relations we use the scattering problem satisfied by a generalisation of the 'squared eigenfunction'. In this generalisation the 'squared eigenfunctions' are compounded from the eigenfunctions for two distinct sets of potentials satisfying the scattering problem. Undoubtedly the same results can be obtained by the related method of 'generalised Wronskians' developed by Calogero (1975) and Calogero and Degasperis (1976a, b) but it seems simpler to proceed directly from the scattering problem itself. This approach has also been applied to the generalised Zakharov–Shabat scheme (Dodd and Bullough 1977). Our results in fact correspond to a discrete version of the results of Calogero (1975) and Calogero and Degasperis (1976b) for the Schrödinger scattering problem.

2. Relationships between the generalised 'squared eigenfunctions' and the scattering data

Consider the self-adjoint operator L which satisfies the scattering problem (Flaschka 1974)

$$(Lu)(n) \equiv a(n-1)u(n-1)+a(n)u(n+1)+b(n)u(n)=\lambda u(n) \tag{2.1}$$

where

$$\lambda = \tfrac{1}{2}(z+z^{-1}).$$

The potentials $a(n)$, $b(n)$ are finite valued and satisfy the boundary conditions $a(n)\to\frac{1}{2}$, $b(n)\to 0$ exponentially as $|n|\to\infty$ and $a(n)>0$ for all n, $n\in Z$ (the integers). We consider that the potentials and eigenfunctions depend smoothly on spatial parameters $y=(y_1,\ldots,y_m)$ as well as a temporal parameter t. For notational simplicity we suppress all parameters until we make specific reference to them. The main features connected with this scattering problem are given in table 1.

In this table $\alpha^{-1}(k)$ and $\beta(k)\alpha^{-1}(k)$ are identified as the transmission and reflection coefficients respectively of the scattered waves from a 'plane wave' incident from the right. The fundamental relations (entry 4) in table 1 are justified by the

Table 1. Scattering theory for $(Lu)(n)\equiv a(n-1)u(n-1)+a(n)u(n+1)+b(n)u(n)=\frac{1}{2}(z+z^{-1})u(n)$.

1. *Spectrum of L*	point spectrum $z_j\in\,]-1,1[$	for $j=1,\ldots,N$	
	continuous spectrum $z=e^{ik}$	$k\in[0,2\pi[$	
2. *Jost functions*	$\phi(n,z)\sim z^{n}$	as $n\to\infty$	
	$\psi(n,z)\sim z^{-n}$	as $n\to-\infty$	
3. *Bound states*	$\sum_{n=-\infty}^{\infty}\zeta^2(n,z_j)=1$		
4. *Fundamental relations*	$\psi(n,k)=\beta(k)\phi(n,k)+\alpha(k)\phi(n,-k)$		$k\in[0,2\pi[$
	$\phi(n,k)=-\beta(-k)\psi(n,k)+\alpha(k)\psi(n,-k)$		
	$\alpha(z)$ is analytic for $\|z\|<1$		
5. *Asymptotic form of fundamental solutions*	$\psi(n,k)\sim\beta(k)\,e^{ink}+\alpha(k)\,e^{-ink}$	$n\to\infty$	
	$\phi(n,k)\sim-\beta(-k)\,e^{-ink}+\alpha(k)\,e^{ink}$	$n\to-\infty$	
	$\zeta(n,z_j)\sim c_j(z_j)^n$	$n\to\infty$	
	c_j are the normalisation constants for the proper eigenfunctions		

pairwise linear independence of the Jost functions on the unit circle. The set

$$S=\{c_j, z_j, \alpha^{-1}(k), \beta(k)\alpha^{-1}(k), k[0, 2\pi[, z_j\in]-1, 1[, j=1, \ldots, N\}$$

constitutes the scattering data for a specific pair of potentials a, b. Knowledge of S is equivalent to the specification of the spectral distribution function for L and therefore allows the unique reconstruction of the potentials a and b. The reconstruction of the potentials from S constitutes the inverse scattering problem and this has been fully investigated by Flaschka (1974), Case and Kac (1973) and Case (1973).

A second pair of potentials $a'(n)$, $b'(n)$ with primed parameters, satisfies equation (2.1) with primed generalised eigenfunctions $u'(n)$. Introduce the definitions

$$v(n)=f(n)g'(n) \qquad w(n)=f(n)g'(n+1) \qquad w'(n)=f(n+1)g'(n) \tag{2.2}$$

where $f(n)$ and $g'(n)$ are generalised eigenfunctions satisfying (2.1) for the unprimed and primed potentials respectively. It is immediate from (2.1) that

$$a(n-1)w(n-1)+a(n)w'(n)+b(n)v(n)=\lambda v(n) \tag{2.3a}$$

$$a'(n-1)w'(n-1)+a'(n)w(n)+b'(n)v(n)=\lambda v(n). \tag{2.3b}$$

From equations (2.3) we obtain the recursion formula

$$\begin{aligned} a(n)w'(n)-a'(n-1)w'(n-1) \\ = h(n)\equiv a'(n)w(n)+(b'(n)-b(n))v(n)-a(n-1)w(n-1). \end{aligned} \tag{2.4}$$

The recursion formula (2.4) leads to the following formula for $w'(n)$:

$$\begin{aligned} w'(n)=&\sum_{j=l}^{n} c(j)h(j)+c(l)[a(l)w'(l)-a'(l)w(l)+a(l-1)w(l-1) \\ &-(b'(l)-b(l))v(l)]+c(n)a'(n)w(n)+c(n)(b'(n)-b(n))v(n) \\ =&\sum_{j=l}^{n-1}\begin{pmatrix} c(j)(b'(j)-b(j)) \\ c(j)(a'^2(j)-a^2(j))a'^{-1}(j)\end{pmatrix}\cdot\begin{pmatrix} v(j) \\ w(j)\end{pmatrix}+c(l)[a(l)w'(l)-a'(l)w(l) \\ &-(b'(l)-b(l))v(l)]+c(n)a'(n)w(n)+c(n)(b'(n)-b(n))v(n) \end{aligned} \tag{2.5a}$$

where

$$c(j)=\frac{1}{a(n)}\prod_{l=j}^{n-1}\frac{a'(l)}{a(l)}, \qquad j<n, \qquad c(n)=\frac{1}{a(n)} \tag{2.5b}$$

and '$\cdot$' is a scalar product between two-component vectors. We have suppressed the index n in $c(j)$ for notational clarity (i.e. for $c(j)$ read $c(n, j)$). Substituting (2.5a) into (2.3a) produces the important relation

$$\begin{aligned} a(n)&\sum_{j=l}^{n-1}\begin{pmatrix} c(j)(b'(j)-b(j)) \\ c(j)(a'^2(j)-a(j)^2)a'^{-1}(j)\end{pmatrix}\cdot\begin{pmatrix} v(j) \\ w(j)\end{pmatrix}+a(n)c(l)[a(l)w'(l)-a'(l)w(l) \\ &-(b'(l)-b(l))v(l)]+b'(n)v(n)+a'(n)w(n)+a(n-1)w(n-1)=\lambda v(n). \end{aligned} \tag{2.6}$$

From the scattering problem (2.1) it is straightforward to obtain a further relation. We begin from the formula

$$\lambda(a'(n)w(n)-a(n-1)w(n-1))$$
$$= s(n) \equiv b(n)a'(n)w(n)+a'(n)a(n)v(n+1)-a(n-1)a'(n-1)v(n-1) - a(n-1)b'(n)w(n-1). \qquad (2.7)$$

Equation (2.7) yields the second important relation

$$\lambda(w(n)-a'(l)d(l)w(l))$$
$$=\sum_{j=l}^{n} s(j)d(j)-d(l)s(l)$$
$$=\sum_{j=l}^{n-1}\left(\frac{d(j)(a^2(j-1)-a'^2(j))}{d(j)a'(j)(b(j)-b'(j+1))}\right)\cdot\left(\frac{v(j)}{w(j)}\right)+d(n)a'(n)a(n)v(n+1)$$
$$+b(n)a'(n)w(n)d(n)+d(n-1)a'(n-1)a(n-1)v(n)$$
$$-d(l-1)a'(l-1)a(l-1)v(l)-d(l)(a'(l)a(l)v(l+1)+b(l)a'(l)w(l)) \qquad (2.8a)$$

where

$$d(l)=\frac{1}{a'(n)}\prod_{j=l}^{n-1}\frac{a(j)}{a'(j)}, \qquad j<n, \qquad d(n)=\frac{1}{a'(n)} \qquad (2.8b)$$

and as for equation (2.5b) we have suppressed the index n in $d(l)$.

Equations (2.6) and (2.8a) are fundamental to the rest of this paper. Letting $l\to-\infty$ and choosing $f=\psi(k)$, $g'=\psi'(k)$ we obtain the eigenvalue problem satisfied by the generalisation of the 'squared eigenfunctions' mentioned in the introduction,

$$\Lambda\left(\frac{v}{w}\right)=\lambda\left(\frac{v}{w}\right). \qquad (2.9a)$$

Here $\left(\frac{v}{w}\right)$ denotes a two-'doubly-infinite'-component column vector (both v and w have a doubly infinite number of comonents since $n\in Z$). The nth components of the left-hand side of equation (2.9a) are given by

$$\left\{\Lambda\left(\frac{v}{w}\right)\right\}(n)=\left[\frac{a(n)\sum_{j=-\infty}^{n-1}\left(\frac{c(j)(b'(j)-b(j))}{c(j)(a'^2(j)-a^2(j)a'^{-1}(j))}\right)\cdot\left(\frac{v(j)}{w(j)}\right)+b'(n)v(n)+a(n-1)w(n-1)+a'(n)w(n)}{\sum_{j=-\infty}^{n-1}\left(\frac{d(j)(a^2(j-1)-a'^2(j))}{d(j)a'(j)(b(j)-b'(j+1))}\right)\cdot\left(\frac{v(j)}{w(j)}\right)+d(n-1)a'(n-1)a(n-1)v(n)+d(n)a'(n)a(n)v(n+1)+d(n)b(n)a'(n)w(n)}\right] \qquad (2.9b)$$

and for this particular case $v(j)=\psi(j,k)\psi'(j,k)$ and $w(j)=\psi(j,k)\psi'(j+1,k)$. The adjoint operator Λ^+ is defined by,

$$\left\{\Lambda^+\left(\frac{p}{q}\right)\right\}(n)=\begin{bmatrix} b'(n)p(n)+d(n-1)a'(n-1)a(n-1)q(n) \\ \quad +d(n-1)a'(n-1)a(n-1)q(n-1)+c(n)(b'(n)-b(n)) \\ \quad \times \sum_{j=n+1}^{\infty} a(j)p(j)+d(n)(a^2(n-1)a'^2(n)) \sum_{j=n+1}^{\infty} q(j) \\ \hline a(n)p(n+1)+a'(n)p(n)+d(n)b(n)a'(n)q(n) \\ \quad +c(n)(a'^2(n)-a^2(n))a'^{-1}(n) \sum_{j=n+1}^{\infty} a(j)p(j) \\ \quad +d(n)a'(n)(b(n)-b'(n+1)) \sum_{j=n+1}^{\infty} q(j) \end{bmatrix} \tag{2.10}$$

where $\left(\frac{p}{q}\right)$ is the adjoint eigenstate to $\left(\frac{v}{w}\right)$.

We next produce formulae connecting the scattering data for the two potentials with the generalised 'squared eigenfunctions'. These are basic to the generalised Bäcklund transformations derived in the next section. In the remainder of this section however we shall use them to determine the variation of the scattering data and specify the class of solvable non-linear partial differential-difference equations. These results are a slight generalisation of the work of Flaschka and McLaughlin (1976).

Our starting point is equation (2.6) in which we allow $n\to\infty$, $l\to-\infty$. Use of table 1 then yields the following formulae for different choices of f and g':

$$\alpha\beta'-\alpha'\beta=\frac{\mathrm{i}}{2\sin k}I(\psi(k),\psi'(k)) \tag{2.11a}$$

$$\alpha'-\prod_{l=-\infty}^{\infty}\frac{a'(l)}{a(l)}\alpha=-\frac{\mathrm{i}}{2\sin k}I(\phi(k),\psi'(k)) \tag{2.11b}$$

$$\alpha-\prod_{l=-\infty}^{\infty}\frac{a'(l)}{a(l)}\alpha'=\frac{\mathrm{i}}{2\sin k}I(\psi(k),\phi'(k)) \tag{2.11c}$$

where

$$I(f,g')=\sum_{j=-\infty}^{\infty}\left(\frac{(b'(j)-b(j))C(j)}{(a'^2(j)-a^2(j))C(j)a'^{-1}(j)}\right)\cdot\left(\frac{f(j)g'(j)}{f(j)g'(j+1)}\right) \tag{2.11d}$$

$$C(j)=\lim_{n\to\infty}\{c(j)\} \qquad \text{and} \qquad k\in\,]0,2\pi[. \tag{2.11e}$$

Analogous formulae for β' and β can be obtained but these are not required. We now introduce the definition

$$\alpha\beta'-\alpha'\beta=\frac{\mathrm{i}}{2\sin k}I(\psi(k),\psi'(k))\stackrel{\text{def}}{=}\Omega(k)(\alpha\beta'\,\mathrm{e}^{\mathrm{i}k}-\alpha'\beta\,\mathrm{e}^{-\mathrm{i}k}). \tag{2.12}$$

In the unsuppressed notation $\Omega(k)$ is

$$\Omega(\lambda,y,t)=h(\lambda,y,t)/l(\lambda,y,t) \tag{2.13}$$

with $\lambda=\frac{1}{2}(\mathrm{e}^{\mathrm{i}k}+\mathrm{e}^{-\mathrm{i}k})$ and h and l are two entire but arbitrary functions of λ. A full discussion of the definition in equation (2.12) is given in § 4. However we now show that the definition enables the trivial integration of the equations defining the evolution of the scattering data when these do not depend on the spatial parameters $\boldsymbol{y}$. In the general case the definition equation (2.12) ensures that the scattering data satisfy *linear* partial differential equations.

We require that in the limit $(a', b') \to (a, b)$ along some path. Equations (2.11*a*), (2.11*b*) and (2.12) become

$$\Delta\alpha^{-1} = \frac{\mathrm{i}}{\sin k} J(\phi(k), \psi(k))\alpha^{-1} \tag{2.14a}$$

$$\Delta\left(\frac{\beta}{\alpha}\right) = \frac{\mathrm{i}}{\sin k} J(\psi(k), \psi(k))\alpha^{-2} = 2\mathrm{i} \sin k\,\Omega(k)\left(\frac{\beta}{\alpha}\right) \tag{2.14b}$$

$$\Delta \equiv \frac{\partial}{\partial t} + \boldsymbol{h}(t, \boldsymbol{y}, \lambda) \cdot \frac{\partial}{\partial \boldsymbol{y}} \tag{2.14c}$$

$$J(f, g) \equiv \sum_{j=-\infty}^{\infty} \left(\frac{\Delta b(j)}{2\Delta a(j)}\right) \cdot \left(\frac{f(j)g(j)}{f(j)g(j+1)}\right). \tag{2.14d}$$

Δ is the general differential operator compatible with the limiting process described above for which the solvable non-linear equations take the form of evolution differential-difference equations when $\boldsymbol{h} \equiv 0$. With this definition of Δ a Lax formulation (Lax 1968) exists provided

$$\Delta(\lambda) = 0 \Rightarrow \Delta z = 0, \qquad z \notin \{-1, 1\}. \tag{2.15}$$

The corresponding evolution of the normalisation 'constants' c_j for the eigenfunctions $\phi_j(n) = \phi(n, z_j)$ where the eigenvalue $z_j \in]-1, 1[$, can be obtained in the following way. From equation (2.1) and the fundamental relations (entry 4) in table 1, we observe that

$$\sum_{j=-\infty}^{\infty} \phi_j(n)\phi(k, n) = -\tfrac{1}{2}\beta_j^{-1} \lim_{n\to\infty}\left(\frac{\alpha(k)z_j^n z^{-n}(z_j - z^{-1})}{\mu - \lambda_j} + \frac{\beta(k)z^n z_j^n(z_j - z)}{\mu - \lambda_j}\right) \tag{2.16}$$

where $\lambda_j = \frac{1}{2}(z_j + z_j^{-1})$, $\mu = \cos k$ and we have used the fact that $\psi_j(n) = \beta_j\phi_j(n)$. From equation (2.16) we conclude that the normalisation constants c_j are defined after letting $\mu \to \lambda_j$ by

$$\sum_{j=-\infty}^{\infty} \phi_j(n)\phi_j(n) = c_j^{-2} = -z_j^{-1}\left(\frac{\beta_j}{\dot{\alpha}_j}\right)^{-1} \tag{2.17a}$$

upon using

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{2z^2}{z^2 - 1}\frac{\mathrm{d}}{\mathrm{d}z} \qquad \text{and} \qquad \dot{\alpha}_j = \left.\frac{\mathrm{d}\alpha}{\mathrm{d}z}\right|_{z=z_j} \tag{2.17b}$$

Consequently

$$\Delta c_j = \mathrm{i}\Delta \lim_{z\to z_j}\left(z\frac{\beta(z)}{\alpha(z)}(z - z_j)\right)^{1/2} = \tfrac{1}{2}c_j[(z_j - z_j^{-1}) + \boldsymbol{h}_{,z_j} \cdot z_{j,\boldsymbol{y}}] \tag{2.18a}$$

and

$$\Delta z_j = 0. \tag{2.18b}$$

Equations (2.14) and (2.18) define the evolution of the scattering data. For the case when the potentials and the eigenvalues do not depend on the parameters y, these equations reduce to the usual form (Flaschka 1974). The corresponding solvable non-linear partial differential-difference equations are obtainable from (2.14b). We jump ahead a little here and use the results in § 3, equations (3.2a) and (3.2d) to write this relation as

$$J(\psi(k), \psi(k)) + \Omega(k) K(\psi(k), \psi(k)) = 0. \tag{2.19}$$

Consequently

$$\sum_{j=-\infty}^{\infty} \left[\left(\frac{\Delta b(j)}{2\Delta a(j)}\right) + \Omega(k)\left(\frac{2(a^2(j-1) - a^2(j))}{2a(j)(b(j) - b(j+1))}\right)\right] \cdot \left(\frac{\psi(j)\psi(j)}{\psi(j)\psi(j+1)}\right) = 0 \tag{2.20a}$$

and the solvable non-linear partial differential-difference equations are

$$\left(\frac{\Delta b}{2\Delta a}\right) + \Omega(\theta)\left(\frac{G}{H}\right) = 0 \tag{2.20b}$$

where

$$\left(\frac{G(j)}{H(j)}\right) \equiv \left(\frac{2(a^2(j-1) - a^2(j))}{2a(j)(b(j) - b(j+1))}\right) \quad \text{and} \quad \theta = \Lambda^+(a' = a, b' = b). \tag{2.20c}$$

Notice that equation (2.20b) is written as a two-doubly-infinite-component vector equation.

3. Generalised Bäcklund transformations

In this section we derive the generalised Bäcklund transformation as well as some other functional relations between *solutions* of the same non-linear partial differential-difference equation mentioned in the introduction. Historically Bäcklund transformations arose from the study of the transformation between surfaces in three-space. The natural generalisation of this concept to n-dimensional space indicates that $n-1$ algebraically distinct relations are needed between the quantities defining surface elements on the two surfaces to constitute a Bäcklund transformation (Dodd and Bullough 1976). However the extensions to more than two independent variables current in the literature and still referred to as Bäcklund transformations arise from the alternative interpretation of a Bäcklund transformation as a transformation between solutions to partial differential equations. In all the cases so far examined the transformation consists of two equations relating the solutions and their partial derivatives.

Toda and Wadati (1975) and Chen and Liu (1975) have obtained the Bäcklund transformation for the Toda lattice by extending the definition to embrace differential-difference equations. These authors show that an algebraic superposition principle exists for solutions to the Toda lattice and use it to calculate the N-soliton solution from the Bäcklund transformation. Although the deeper mathematical significance of the existence of a Bäcklund transformation with regards to algebraic properties of the original equation is being developed in the continuous case (cf Wahlquist and Estabrook 1975, Dodd and Gibbon 1977) for the discrete case this still remains to be investigated.

We re-derive the Bäcklund transformation for the Toda lattice, *entirely* from the scattering problem, as the simplest example from the general Bäcklund transformation formula.

The general Bäcklund transformations are obtained from the definition in (2.12),

$$I(\psi(k), \psi'(k)) = -\Omega(k)[\alpha\beta'(e^{2ik}-1)+\alpha'\beta(e^{-2ik}-1]. \tag{3.1}$$

From equation (2.8a) by letting $n\to\infty$, $l\to-\infty$ and taking suitable choices of f and g' further relations between the Jost functions and the scattering data are produced:

$$\alpha'\beta(e^{-2ik}-1)+\alpha\beta'(e^{2ik}-1)=K(\psi(k), \psi'(k)) \tag{3.2a}$$

$$(1-e^{2ik})\alpha'^{-1}(k)\left(\prod_{l=-\infty}^{\infty}\frac{a(l)}{a'(l)}\alpha'(k)-\alpha(k)-\beta'(-k)(\alpha(k)\beta'(k)-\alpha'(k)\beta(k)\,e^{-2ik})\right)$$
$$=K(\psi(k), \psi'(-k)) \tag{3.2b}$$

$$(1-e^{2ik})\alpha^{-1}(k)\left[\left(\alpha'-\prod_{l=-\infty}^{\infty}\frac{a(l)}{a'(l)}\alpha(k)\right)e^{-2ik}-\beta(-k)(\alpha(k)\beta(-k)-\alpha'(k)\beta(k)\,e^{-2ik})\right]$$
$$=K(\psi(k), \psi'(k)) \tag{3.2c}$$

where

$$K(f,g)\equiv 8\sum_{j=-\infty}^{\infty}\begin{pmatrix}C^{-1}(j)(a^2(j-1)-a'^2(j))\\ \hline C^{-1}(j)a'(j)(b(j)-b'(j+1))\end{pmatrix}\cdot\begin{pmatrix}f(j)g'(j)\\ \hline f(j)g'(j+1)\end{pmatrix}. \tag{3.2d}$$

In obtaining the equations (3.2b), (3.2c) use has been made of the relationship (conservation of probability),

$$\alpha(-k)\alpha(k)=1+\beta(-k)\beta(k) \tag{3.3}$$

which is obtained directly from the fundamental relations (entry 4) in table 1. Equation (3.1) can therefore be written as

$$\sum_{j=-\infty}^{\infty}\begin{pmatrix}A(j)\\ \hline B(j)\end{pmatrix}\cdot\begin{pmatrix}f(j)g'(j)\\ \hline f(j)g'(j+1)\end{pmatrix}=0 \tag{3.4a}$$

where

$$\begin{pmatrix}A(j)\\ \hline B(j)\end{pmatrix}\equiv l(k)\begin{pmatrix}P(j)\\ \hline Q(j)\end{pmatrix}+8h(k)\begin{pmatrix}R(j)\\ \hline S(j)\end{pmatrix}$$
$$\equiv l(k)\begin{pmatrix}(b'(j)-b(j))C(j)\\ \hline (a'^2(j)-a^2(j))C(j)a'^{-1}(j)\end{pmatrix}+8h(k)\begin{pmatrix}C^{-1}(j)(a^2(j-1)-a'^2(j))\\ \hline C^{-1}(j)(b(j)-b'(j+1))a'(j)\end{pmatrix}. \tag{3.4b}$$

The adjoint operator Λ^+, equation (2.10), and the definition of $\Omega(k)$ as a ratio of entire functions (2.13) enables the following relationship to be obtained from (3.4a),

$$l(\Lambda^+)\begin{pmatrix}P\\ \hline Q\end{pmatrix}+8h(\Lambda^+)\begin{pmatrix}R\\ \hline S\end{pmatrix}=0. \tag{3.5}$$

Equation (3.5) is a functional relation connecting pairs of potentials (a', b'), (a, b) of the scattering problem (2.1) and therefore constitutes a transformation between two different solutions to any one of the solvable non-linear partial differential-difference equations defined by (2.20b). Equation (3.5) is the general Bäcklund transformation formula for equations (2.20*b*). A fuller discussion of the general transformation defined by (3.5) will be given elsewhere. Here we merely produce the simplest member of the class of transformations by putting $\Omega(k)=p$ – a constant. In this case equation (3.5) is

$$b(n)-b'(n)=2p\prod_{l=n}^{\infty}\left(\frac{a(l)}{a'(l)}\right)^2(a^2(n-1)-a'^2(n)) \tag{3.6a}$$

$$b'(n+1)-b(n)=\frac{1}{2}p^{-1}\prod_{l=n}^{\infty}\left(\frac{a'(l)}{a(l)}\right)^2(a'^2(n)-a^2(n))a'^{-2}(n). \tag{3.6b}$$

This Bäcklund transformation is easily brought into the more familiar form (Toda and Wadati 1975, Wadati 1976) by the substitution for the Toda lattice, $b(n)=-\frac{1}{2}\dot{Q}(n-1)$, $2\ln(2a(n))=-(Q(n)-Q(n-1))$ which quickly yields

$$\prod_{j=n}^{\infty}\left(\frac{a(j)}{a'(j)}\right)^2=\exp(Q(n-1)-Q'(n-1)+Q'(\infty)-Q(\infty))$$

so that equations (3.6) become

$$\dot{Q}'(n-1)-\dot{Q}(n-1)$$
$$=A\{\exp[-(Q'(n-1)-Q(n-2))]-\exp[-(Q'(n)-Q(n-1))]\} \tag{3.7a}$$

$$\dot{Q}(n-1)-\dot{Q}'(n)=A^{-1}\{\exp[-(Q(n-1)-Q'(n-1))]-\exp[-(Q(n)-Q'(n))]\} \tag{3.7b}$$

where $A=p\exp(Q'(\infty)-Q(\infty))$. The change in the scattering data corresponding to the Bäcklund transformation (3.5) is obtained from equations (2.11), (2.12) and (3.2). Thus we obtain the formulae

$$\left[\alpha'-\prod_{j=-\infty}^{\infty}\left(\frac{a'(j)}{a(j)}\right)\alpha\right]l(k)+\mathrm{e}^{-\mathrm{i}k}\left[\prod_{j=-\infty}^{\infty}\left(\frac{a(j)}{a'(j)}\right)\alpha-\alpha'\right]h(k)=\alpha(k)M(k) \tag{3.8a}$$

$$\left[\alpha-\prod_{j=-\infty}^{\infty}\left(\frac{a'(j)}{a(j)}\right)\alpha'\right]l(k)+\mathrm{e}^{\mathrm{i}k}\left[\prod_{j=-\infty}^{\infty}\left(\frac{a(j)}{a'(j)}\right)\alpha'-\alpha\right]h(k)=\alpha'(k)N(k) \tag{3.8b}$$

$$\frac{\beta}{\alpha}=\mathrm{e}^{\mathrm{i}k}\left(\frac{l-h\,\mathrm{e}^{\mathrm{i}k}}{l\,\mathrm{e}^{\mathrm{i}k}-h}\right)\frac{\beta'}{\alpha'} \tag{3.8c}$$

where

$$M(k)\equiv-\frac{\mathrm{i}}{2\sin k}[l(k)I(\psi(-k),\psi'(k))+h(k)K(\psi(-k),\psi'(k))]$$
$$=-\frac{\mathrm{i}}{2\sin k}\sum_{j=-\infty}^{\infty}(\Lambda^+-\lambda)^{-1}\left[\left(\frac{A(j)}{B(j)}\right)\cdot\left(\frac{E(j,k)}{F(j,k)}\right)\right] \tag{3.8d}$$

$$N(k)\equiv\frac{\mathrm{i}}{2\sin k}[l(k)I(\psi(k),\psi'(k))+h(k)K(\psi(k),\psi'(-k))]$$
$$=\frac{\mathrm{i}}{2\sin k}\sum_{j=-\infty}^{\infty}(\Lambda^+-\lambda)^{-1}\left[\left(\frac{A(j)}{B(j)}\right)\cdot\left(\frac{E(j,-k)}{F(j,-k)}\right)\right] \tag{3.8e}$$

and $\left(\frac{A}{B}\right)$ is defined by (3.4*b*) and $\left(\frac{E(k)}{F(k)}\right)$ by

$$\left(\frac{E(j,k)}{F(j,k)}\right)=\left(\frac{\prod_{l=-\infty}^{j-1}(a'(l)/a(l))\mathrm{i}\sin k}{\frac{1}{4}a'^{-1}(j)\prod_{l=-\infty}^{j-1}(a(l)/a'(l))(\mathrm{e}^{-2\mathrm{i}k}-1)}\right). \tag{3.8f}$$

For the case given by (3.6), (3.7), $\Omega(k)=p$, equations (3.8) become, using (3.5),

$$\frac{\beta}{\alpha}=\frac{1-p\,\mathrm{e}^{\mathrm{i}k}}{\mathrm{e}^{\mathrm{i}k}-p}\,\mathrm{e}^{\mathrm{i}k}\frac{\beta'}{\alpha'} \tag{3.9a}$$

$$\alpha^{-1}=\frac{1-p\,\mathrm{e}^{\mathrm{i}k}}{p-\mathrm{e}^{\mathrm{i}k}}\alpha'^{-1} \tag{3.9b}$$

and

$$\prod_{j=-\infty}^{\infty}\frac{a'(j)}{a(j)}=|p|. \tag{3.9c}$$

We take the modulus in (3.9c) because $a(j)>0$ for all $j\in Z$. We note that (3.9b) is in agreement with the Poisson–Jensen identity (Flaschka 1974) when α contains an additional eigenvalue to α' (i.e. an extra soliton).

$$\ln\alpha(z)=-\ln\alpha(0)+\frac{1}{\pi}\int_0^{\pi}\mathrm{d}k\left[\ln(|\alpha(k)|^2)\left(\frac{\mathrm{e}^{\mathrm{i}k}}{\mathrm{e}^{\mathrm{i}k}-z}\right)\right]$$
$$+\sum_{j=1}^{N}\ln\left(\frac{z-z_j}{z-z_j^{-1}}\right)+\ln\left(\frac{z-p}{z-p^{-1}}\right)=\ln\alpha'(z)+\ln\left(\frac{z-p}{zp-1}\right)\qquad |z|<1. \tag{3.10}$$

Equations (3.9) indicate that the addition of a soliton is always accompanied by a change in phase of $\beta(k)$, thus $\beta(k)=-\mathrm{e}^{\mathrm{i}k}\beta'(k)$. Consider the particular class of evolution difference equations defined by

$$\frac{\partial}{\partial t}\left(\frac{b}{2a}\right)+\phi(\theta)\left(\frac{G}{H}\right)=0, \tag{3.11}$$

see equations (2.20b) and (2.20c) for the definitions of G and H. Then equations (2.14) imply that

$$\frac{\partial}{\partial t}\left(\frac{\beta}{\alpha}\right)=2\mathrm{i}\sin k\phi(k)\left(\frac{\beta}{\alpha}\right), \tag{3.12a}$$

where $\phi(k)=\phi(\mathrm{i}\sin k)$, which can be integrated to yield,

$$\frac{\beta}{\alpha}(t)=\frac{\beta}{\alpha}(t')\exp[2\mathrm{i}\sin k\phi(k)(t-t')]. \tag{3.12b}$$

From equation (3.9a) we see that this corresponds to a generalised Bäcklund transformation provided

$$\frac{\beta}{\alpha}(t)=\mathrm{e}^{\mathrm{i}k}\left(\frac{1-\mathrm{e}^{\mathrm{i}k}\Omega(k)}{\mathrm{e}^{\mathrm{i}k}-\Omega(k)}\right)\frac{\beta}{\alpha}(t') \tag{3.12c}$$

where we have defined $(\beta/\alpha)(t')=\beta'\alpha'$. Equating equations (3.12b) and (3.12c) and solving for Ω we obtain

$$\Omega(k)=\frac{\mathrm{e}^{\mathrm{i}k}\{\exp[2\mathrm{i}\sin k\phi(k)(t-t')]-1\}}{\exp[2\mathrm{i}\sin k\phi(k)(t-t')]-\mathrm{e}^{2\mathrm{i}k}}. \tag{3.13}$$

The corresponding generalised Bäcklund transformation is from equation (3.5),

$$8[\lambda^+ + (1+\lambda^{+2})^{1/2}]\{\exp[2(t-t')\lambda^+\phi(\lambda^+)]-1\}\left(\frac{R}{S}\right) + [\![\exp[2(t-t')\lambda^+\phi(\lambda^+)]$$

$$-\{\lambda^+ + [1+(1+\lambda^{+2})^{1/2}]^2\}]\!]\left(\frac{P}{Q}\right) = 0, \qquad (3.14a)$$

where

$$\lambda^+ = \Lambda^+(a' = a(t'), a = a(t)). \qquad (3.14b)$$

In the limit, $t' \leftrightarrow t$ we recover equation (3.11) from equation (3.14a). Equation (3.14) is therefore a generalised Bäcklund transformation which connects the same solution to a given differential-difference equation at two different times.

4. Conclusion

Besides obtaining the generalised Bäcklund transformations for the partial differential-difference equations related to the discretised Schrödinger equation we have also outlined a general method for obtaining these transformations which provides an alternative to the generalised Wronskian technique of Calogero and Degasperis (e.g. Calogero and Degasperis 1975, 1976a, b). In this conclusion we sketch this method for linear differential operators on the real line and expect that it is applicable to a wide class of inverse methods (e.g. the N-channel Schrödinger problem and cf Dodd and Bullough 1977). Let

$$L(V)w = \lambda w \qquad (4.1)$$

represent a scattering problem where L is a linear differential operator depending on the potentials V and w is an eigenvector (e.g. $V = (q_1, \ldots, q_n)$, w and $n \times 1$ column vector). Let the boundary conditions be $x \to \pm\infty$, $V \to 0$ and the corresponding fundamental matrix solutions defined by these boundary conditions be denoted Ψ, Φ, and let $\Phi = \Psi A$ define the scattering matrix A.

(i) Let Φ, Φ' be two fundamental matrix solutions to (4.1) and let V, V' denote the corresponding potential functions. Then from equation (4.1) determine the operator equation satisfied by the generalised squared eigenfunctions $W(\Phi, \Phi') = \Phi^{-1}\Phi'$,

$$\Lambda(V, V')W(\Phi, \Phi') = \lambda W(\Phi, \Phi'). \qquad (4.2)$$

(ii) Using the matrix functions Φ, Φ' in (4.1) and subtracting the resulting equations and then integrating one can obtain a relationship between the scattering data and an inner product. This inner product is between quadratic products compounded from the fundamental solutions, and functions of the potentials. We write this as

$$A^{-1}(A' - A) = \langle M(V, V'), W(\Phi, \Phi')\rangle. \qquad (4.3)$$

(iii) From (4.1) it is possible to obtain a second relationship between the transmission and reflection coefficients, the potentials and $W(\Phi, \Phi')$,

$$A^{-1}K(A, A') = \langle N(V, V'), W(\Phi, \Phi')\rangle. \qquad (4.4)$$

(iv) Consequently if $\Omega(\lambda)$ is an entire function of λ, equations (4.3) and (4.4) imply that

$$A' - A = \Omega(\lambda)K(A, A') \tag{4.5a}$$

and

$$\langle M(V, V') - \Omega(\lambda)N(V, V'), W(\Phi, \Phi')\rangle = 0 \tag{4.5b}$$

for a special set of potentials V, V'. In particular using (4.3), potentials which are related by

$$M(V, V') - \Omega(\Lambda^{+})N(V, V') = 0 \tag{4.6}$$

provide a sufficient non-trivial condition for (4.5*b*) to be satisfied. We term equation (4.6) the generalised Bäcklund transformations for the associated solvable equations, (really only one half of the transformation, the other half can be recovered from the equation itself and (4.6)). Equation (4.5*a*) and similar equations will express the corresponding change in the scattering data when one transforms using (4.6) between the sets of potentials.

The above outline is only a sketch and not to be taken too literally. Modifications will be necessary depending upon the specific scattering problem.

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